ON NECESSARY AND SUFFICIENT CONDITIONS FOR THE OSCILLATION OF ALL SOLUTIONS OF ANOTHER SPECIAL CASE OF $x' + \sum_{i=1}^{n} p_i x(t-\tau_i) = 0$

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Abstract
In this paper, we establish that all solutions of $x'(t) + p_1x(t-\tau_1) + p_2x(t-4\tau_1) = 0$ are Oscillatory if and only if $\frac{1}{\tau_1} log\xi + p_1\xi + p_2\xi^4 > 0$, where $\xi = \frac{-k^{\frac{1}{2}}+\sqrt{k-2U}}{2}$ and $u = \left[\frac{k^{\frac{2}{3}}-q}{k^{\frac{1}{3}}}\right]$, p_1 and $p_2 > 0$ and τ_1 is the delay,

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Introduction

The retarded differential equations provide mathematical models for physical systems in which the rate of change of the system depends not only on its present stage, but also on its past history. Such equations appear in biology, control theory, ecology, number theory, physics, etc. The Oscillation Theory of retarded differential equations has been extensively discussed for example Tramov (1975), Ladde (1977), Ladas (1979), Kusano (1982) Hunt and Yorke (1984). These authors concluded that is possible to have non-trivial oscillatory solutions for retarded differential equations under appropriate hypothesis.
Ladas and Stavroulakis (1982a) studied the solution to the scalar differential delay equation

$$x' = -px(t-\tau) \qquad \dots (1.0)$$

where p and τ are positive constants. They concluded that a non-zero solution to equation (1.0) oscillates about 0 if and only if

$$\tau p = \frac{1}{e} \qquad \dots(2.0)$$

τp is called the "toque" for equation(1.0) in a sense p represents the magnitude of a force and τ represents from how far away (in time rather than space) this force is applied and e is an exponential. Ladas and Stavroulakis (1982b) and Ladas *et al* (1983) studied an extension of equation (1.0) to

$$x' + \sum_{i=1}^{n} p_i x\big(t-\tau_i(t)\big) = 0 \qquad \dots (3.0)$$

Where τ_i and p_i are positive constants. Hunt and Yorke (1984) further extended this work to

$$x' + \sum_{i=1}^{n} p_i(t) x\big(t-\tau_i(t)\big) = 0 \qquad \dots (4.0)$$

where τ_i and p_i are continuous and positive-valued on $[0,\infty)$.

Arino *et al* (1984) used the method of characteristic equation to study the oscillation of equation (4) to obtained a more explicit sufficient condition and gave a necessary and sufficient condition on the 2n-parameter family $(p_i, \tau_i), i = 1,2,\dots,n, p_i > 0, \tau_i > 0$.
Grove *et al* (1988) established sufficient conditions for the oscillations of the retarded equations

$$x'(t) + px(t-\tau) - qx(t-\sigma) = 0, \qquad \dots(5.0)$$

with positive and negative coefficients and nonnegative delay differential equations

$$\frac{d^n}{dx^n}[x(t) - px(t-\tau)] + qx(t-\sigma) = 0, \quad \dots(6.0)$$

where n is odd and $0 \le p \le 1$, $q \ge 0$ and $\tau, \sigma \ge 0$. The results of Grove et al (1988) are substantial improvement of known results in Arino *et al* (1987), Gÿori (1989) and Ladas and Sficas (1986).
Gÿori *et al* (1989) considered the differential equation with deviating arguments

$$x'(t) + p[x(t-\sigma_1) + x(t-\sigma_2)] = 0 \qquad \dots(7.0)$$

where p, σ_1, σ_2 are real numbers and prove that every unbounded solution of equation (7.0) oscillate if and only if the characteristic equation given by

$$\lambda + p\big(e^{-\lambda\sigma_1} - e^{-\lambda\sigma_2}\big) = 0 \qquad \dots(8.0)$$

has no positive roots and 0 is a simple root of equation (8.0). Gÿori *et al* (1990) used Laplace transform to obtained sufficient conditions for the oscillation of all solutions of the delay differential equation

$x' + \sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)) = f(t) \quad t \geq$ …(9.0)

Where $\in [0,\infty)$ and $p_i \in (-\infty,\infty)$, $\tau_i \in [0,\infty)$ for $i = 1,2,3,\dots,n$. Gÿori *et al* (1990) relied on a result obtained by Widder (1971) about the abscissa of convergence of the Laplace transform of a non-negative function. The results obtained are applicable when, the coefficients p_i are positive and the function $f(t)$ is a finite linear combination of sins and cosines.

Gÿori and Ladas (1991) obtained a necessary and sufficient condition for the oscillation of all solution of the linear autonomous delay differential equation

$x'(t) = \sum_{i=1}^{n} p_i x(t - \tau_i) = 0$ …(10.0)

Where the coefficients p_i are periodic functions with common period and the delays are constants and multiples of these periods.

Philos (1991) established a necessary and sufficient condition for the oscillation of all solutions of first order linear delay differential equations in which the coefficients are periodic functions with a common period and the delays are constants and multiples of this period.

Arino *et al* (1984) remarked that "we could find a necessary and sufficient condition in the form of a single inequality for $\tau_2 = 3\tau_1$ or $2\tau_2 = 3\tau_1$. In both cases the condition can be obtained by solving a third-order polynomial equation; but, of course, this is impossible for $\tau_2 = k\tau_1$ with k being an integer greater than 4. In any case, these inequalities are not easily tractable, therefore it is of a practical interest to be able to find sufficient conditions".

Sesay *et al* (2005) extend the work of Arino *et al* (1984) for a special case of equation (4.0) for $n = 2, \tau_2 = 3\tau_1$, and obtained a necessary and sufficient condition for the oscillation of the equation of the form

$x'(t) + p_1 x(t - \tau_1) + p_2 x(t - 3\tau_1) = 0$ … (11)

Öğûnmez and Öclan (2013) studied the oscillatory behaviour of odd order delay differential equations and extract necessary and sufficient conditions for the oscillation of all solutions of system of differential equation of the form

$x^{(m)}(t) + px(t - \tau) = 0$ …(12.0)

where $p \in \mathbb{R}^{sXs}, \tau \in \mathbb{R}^{+}$ and m is an odd positive integer. Also Öğûnmez and Öclan (2013) obtained sufficient conditions for the oscillation of all solutions of the system of differential equation

$x^{(m)}(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) = 0$ …(13.0)

where $p_i \in \mathbb{R}^{sXs}, \tau_i \in \mathbb{R}^{+}$ for I =1,2,3,…, n and m is an odd positive integer. For an sXs matrix p.

Sesay *et al* (2005) posed some problems for the oscillation of all solutions of equation (4.0) for (i) $n = 3, \tau_2 = 3\tau_1$ and (ii) $n = 4, \tau_2 = 4\tau_1$

The aim of this paper is to obtain a necessary and sufficient condition for the oscillation of a special case of equation (4.0) by extending the work of Arino *et al* (1984), Bamaina (2004) and Sesay *et al* (2005).

Result

Proposition. A necessary and sufficient condition for all solutions of the equation

$x'(t) + p_1 x(t - \tau_1) + p_2 x(t - 4\tau_1) = 0$ …(14.0)

is that the characteristic equation

$q(\lambda) = -\lambda + p_1 e^{\lambda\tau_1} + p_2 e^{4\lambda\tau_1}$ …(15.0)

where p_1 and $p_1 > 0$ and τ_1 is the delay, λ is the unique solution of

$4p_2\tau_1 e^{4\lambda\tau_1} + p_1\tau_1 e^{\lambda\tau_1} - 1 = 0$ … (16.0)

Proof. Let every solution of equation (14.0) oscillate, and then if equation (15.0) has no real root we proceed to show that every solution of equation (14.0) oscillates.

Differentiating (15) with respect to λ gives

$q'(\lambda) = -1 + 4p_2\tau_1 e^{4\lambda\tau_1} + p_1\tau_1 e^{\lambda\tau_1}$ …(17.0)

Looking for the unique root of $q'(\lambda)$, we set $x = e^{\lambda\tau_1}$ then equation (17.0) reduces to the following polynomial equation of degree four

$x^4 + \frac{p_1 x}{4p_2} - \frac{1}{4p_1\tau_1} = 0$ …(18.0)

The above equation (18.0) is equivalent to the following polynomial of degree 4

$x^4 + px^2 + qx + r = 0, with\ p$ …(19.0)

equation (19.0) can be written as

$(x^2 + ax + b)(x^2 - ax + c) = 0$ … (20.0)

where $b = \frac{1}{2}\left(p + a^2 - \frac{q}{a}\right)$, $c = \frac{1}{2}\left(p + a^2 + \frac{q}{a}\right)$, $bc = r$

,i.e.$\frac{1}{4}\left(p + a^2 - \frac{q}{a}\right)\left(p + a^2 - \frac{q}{a}\right) = r$ ...(21.0)

Expanding equation (21.0) and Comparing with equation (18.0) we have $p = 0, q = \frac{p_1}{4p_2}$, $r = -\frac{1}{4p_1\tau_1}$. then, we obtain the following equation

$$a^6 + \frac{1}{p_2\tau_1}a^2 - \frac{p_1^2}{16p_2^2} = 0 \quad \text{....(22.0)}$$

and letting $z = a^2$ the equation becomes

$$z^3 + \frac{1}{p_2\tau_1}z - \frac{p_1^2}{16p_2^2} = 0 \quad \text{... (23.0)}$$

We assume that $4\left(\frac{1}{p_2\tau_1}\right)^3 + 27\left(-\frac{p_1^2}{16p_2^2}\right)^2 > 0$ and let $z = rsinh\theta$ and

$$4sinh^3\theta + 3sinh\theta - sinh3\theta = 0 \quad \text{... (24.0)}$$

Then equation (23.0) becomes

$$r^3sinh^3\theta + \frac{rsinh\theta}{p_2\tau_1} - \frac{p_1^2}{16p_2^2}` = 0 \quad \text{...(25.0)}$$

Comparing equation (24.0) and (25.0), we obtain $\frac{r^3}{4} = \frac{r}{3p_2\tau_1} = -\frac{p_1^2}{16p_2^2 sinh3\theta}$ and from the first two equations, we get $r = 2\left(\frac{1}{3p_2\tau_1}\right)^{\frac{1}{2}}$ and from the last two equations, we also obtain

$$sinh3\theta = \frac{3p_1^2\tau_1}{32\left(\frac{1}{3p_2\tau_1}\right)^{\frac{1}{2}}p_2} \quad \text{.... (26.0)}$$

This gives

$$e^{3\theta} - e^{-3\theta} = \frac{3p_1^2\tau_1}{32\left(\frac{1}{3p_2\tau_1}\right)^{\frac{1}{2}}p_2} \quad \text{...(27.0)}$$

Multiplying both sides of equation (27.0) by $e^{3\theta}$, we have

$$\left(e^{3\theta}\right)^2 - \frac{3p_1^2\tau_1}{32\left(\frac{1}{3p_2\tau_1}\right)^{\frac{1}{2}}p_2}e^{3\theta} - 1 = 0 \quad \text{...(28.0)}$$

Its positive root is

$e^{3\theta} = \frac{\gamma + [\gamma^2+4]^{\frac{1}{2}}}{2}$ where $\gamma = \frac{3p_1^2\tau_1}{32\left(\frac{1}{3p_2\tau_1}\right)^{\frac{1}{2}}p_2}$.

The three values of $e^{3\theta}$ are

$$e^{3\theta} = \left[\frac{\gamma + [\gamma^2+4]^{\frac{1}{2}}}{2}\right]e^{2\pi j}, \left[\frac{\gamma + [\gamma^2+4]^{\frac{1}{2}}}{2}\right]e^{4\pi j}, \left[\frac{\gamma + [\gamma^2+4]^{\frac{1}{2}}}{2}\right]e^{6\pi j}$$

We evaluate to get 3 values of e^{θ} and $e^{-\theta}$.

Then from $2sinh\theta = e^{\theta} - e^{-\theta}$ we have

$$sinh\theta = \frac{1}{2}\left[\left[\frac{\gamma + [\gamma^2+4]^{\frac{1}{2}}}{2}\right]^{\frac{1}{3}} - \left[\frac{\gamma + [\gamma^2+4]^{\frac{1}{2}}}{2}\right]^{-\frac{1}{3}}\right] \quad \text{...(29.0)}$$

From $z = rsinh\theta$ we get three roots of z and two of which are complex conjugate of one another, while the other one is real, that is,

$$z = m^{\frac{1}{2}}\frac{1}{2}\left[l^{\frac{1}{3}} - l^{-\frac{1}{3}}\right] \quad \text{...(30.0)}$$

From which $z = a^2$ after making appropriate substitution, we obtain

$$a = \left[m^{\frac{1}{2}}\left[\frac{l^{\frac{2}{3}} - 1}{l^{\frac{1}{3}}}\right]\right]^{\frac{1}{2}} \quad \text{... (31.0)}$$

where $l = \frac{\gamma + (\gamma+4)^{\frac{1}{2}}}{2}$, $m = \frac{1}{3p_2\tau_1}$, $\gamma = \frac{3p_1^2\tau_1}{16\left(\frac{1}{3p_2\tau_1}\right)^{\frac{1}{2}}p_2}$,

$$q = \frac{p_1}{4p_2}, \quad b = \frac{1}{2}\left[\frac{\left[m^{\frac{1}{2}}\left(\frac{l^{\frac{2}{3}}-1}{l^{\frac{1}{3}}}\right)\right]^{\frac{3}{2}} - q}{\left[m^{\frac{1}{2}}\left(\frac{l^{\frac{2}{3}}-1}{l^{\frac{1}{3}}}\right)\right]^{\frac{3}{2}}}\right] \quad \text{and}$$

$$c = \frac{1}{2}\left[\frac{\left[m^{\frac{1}{2}}\left(\frac{l^{\frac{2}{3}}-1}{l^{\frac{1}{3}}}\right)\right]^{\frac{3}{2}} - q}{\left[m^{\frac{1}{2}}\left(\frac{l^{\frac{2}{3}}-1}{l^{\frac{1}{3}}}\right)\right]^{\frac{3}{2}}}\right]$$

Now from $(x^2 + ax + b)(x^2 - ax + c) = 0$ we have either $(x^2 + ax + b) = 0$ or $(x^2 - ax + c) = 0$. Letting $k = \left[m^{\frac{1}{2}}\left(\frac{l^{\frac{2}{3}}-1}{l^{\frac{1}{3}}}\right)\right]$, the above two quadratic equations will be reduce to

$$x^2 + k^{\frac{1}{2}}x + \frac{1}{2}\left(\frac{k^{\frac{2}{3}} - q}{k^{\frac{1}{3}}}\right) = 0 \quad \text{... (32.0)}$$

or

$$x^2 - k^{\frac{1}{2}}x + \frac{1}{2}\left(\frac{k^{\frac{2}{3}} + q}{k^{\frac{1}{3}}}\right) = 0 \quad \text{...(33.0)}$$

Solving for equation (32.0), we obtain the positive root as $x = \frac{-k^{\frac{1}{2}} + \sqrt{k - 2U}}{2}$ which gives

$$\lambda = \frac{1}{\tau_1}log\left(\frac{-k^{\frac{1}{2}} + \sqrt{k - 2U}}{2}\right) \quad \text{... (34.0)}$$

While equation (33.0) the positive root is $x = \frac{-k^{\frac{1}{2}}+\sqrt{k-2V}}{2}$ which gives

$$\lambda_1 = \frac{1}{\tau_1} log\left(\frac{-k^{\frac{1}{2}}+\sqrt{k-2V}}{2}\right) \qquad \ldots(35.0)$$

We now substitute for λ using equation (32.0) in (15.0) to obtain the following inequality as a necessary and sufficient condition for the oscillation of all solution of equation (14.0) as follows

$$\frac{1}{\tau_1} log\left(\frac{-k^{\frac{1}{2}}+\sqrt{k-2U}}{2}\right) + p_1\left(\frac{-k^{\frac{1}{2}}+\sqrt{k-2U}}{2}\right) + p_2\left(\frac{-k^{\frac{1}{2}}+\sqrt{k-2U}}{2}\right)^4 > 0 \qquad \ldots(36.0)$$

This inequality can further be reduced to $\frac{1}{\tau_1} log\xi + p_1\xi + p_2\xi^4 > 0$. where $\xi = \frac{-k^{\frac{1}{2}}+\sqrt{k-2U}}{2}$ and $u = \left[\frac{k^{\frac{2}{3}}-q}{k^{\frac{1}{3}}}\right]$

Also when we substitute for λ_1 using equation (35.0) in (15.0) to obtain the following inequality as a necessary and sufficient condition for the oscillation of all solution of equation (12.0) as follows

$$\frac{1}{\tau_1} log\left(\frac{-k^{\frac{1}{2}}+\sqrt{k-2V}}{2}\right) + p_1\left(\frac{-k^{\frac{1}{2}}+\sqrt{k-2V}}{2}\right) + p_2\left(\frac{-k^{\frac{1}{2}}+\sqrt{k-2V}}{2}\right)^4 > 0 \qquad \ldots(37.0)$$

Examples

We shall give some examples to illustrate our result that is inequality (36.0):

Example 3.01

$$y'(t) + 7y(t-2) + 2y(t-8) = 0$$

$p_1 = 7, p_2 = 2, \tau_1 = 2, \tau_2 = 8$ and

$\gamma = \frac{3p_1^2\tau_1}{16\left(\frac{1}{3p_2\tau_1}\right)p_2} = 31.826,\ l = \frac{\gamma+(\gamma+4)^{\frac{1}{2}}}{2} = 4.993$

$m = \frac{1}{3p_2\tau_1} = 0.0831,\ q = \frac{p_1}{4p_2} = 0.875,\ k = m^{\frac{1}{2}}\left(\frac{l^{\frac{2}{3}}-1}{l^{\frac{1}{3}}}\right) = 0.324,\ u = \left(\frac{k^{\frac{2}{3}}-q}{k^{\frac{1}{3}}}\right) = -0.588$ and

$\xi = \frac{-k^{\frac{1}{2}}+\sqrt{k-2U}}{2} = 0.328.$

Then the above example satisfies equation (36.0) which is a necessary and sufficient condition for the oscillation of all solution of equation (14.0).

Example 3.02

$y'(t) + \frac{1}{10}y(t-1) + 3y(t-4) = 0$

$p_1 = \frac{1}{10}, p_2 = 3, \tau_1 = 1, \tau_2 = 4,$

$\gamma = \frac{3p_1^2\tau_1}{16\left(\frac{1}{3p_2\tau_1}\right)p_2} = 0.00563,\ l = \frac{\gamma+(\gamma+4)^{\frac{1}{2}}}{2} = 1.0035$

$m = \frac{1}{3p_2\tau_1} 0.111,\ q = \frac{p_1}{4p_2} = 0.0083$, $k = m^{\frac{1}{2}}\left(\frac{l^{\frac{2}{3}}-1}{l^{\frac{1}{3}}}\right) = 0.00077,\ u = \left(\frac{k^{\frac{2}{3}}-q}{k^{\frac{1}{3}}}\right) = 1.09775$ and

$\xi = \frac{-k^{\frac{1}{2}}+\sqrt{k-2U}}{2} = complex\ form$ This example does not satisfy equation (36.0) which is a necessary and sufficient condition for the oscillation of all solution of equation (14.0).

Example 3.03

$$y'(t) + \frac{1}{4}y\left(t-\frac{1}{4}\right) + \frac{1}{3}y(t-1) = 0$$

$p_1 = \frac{1}{4}, p_2 = \frac{1}{3}, \tau_1 = \frac{1}{4}, \tau_2 = 1$ and

$\gamma = \frac{3p_1^2\tau_1}{16\left(\frac{1}{3p_2\tau_1}\right)p_2} = 0.0044,\ l = \frac{\gamma+(\gamma+4)^{\frac{1}{2}}}{2} = 1.003$

$m = \frac{1}{3p_2\tau_1} = 0.0831,\ q = \frac{p_1}{4p_2} = 0.875,$

This example does not satisfy equation (36.0) which is a necessary and sufficient condition for the oscillation of all solution of equation (14.0).

Discussion of Result

According to Gÿori and Ladas (1991), each of the following conditions:

i) $\sum_{i=1}^{n} p_i\tau_i > \frac{1}{e}$

ii) $\left(\prod_{i=1}^{n} p_i\right)^{\frac{1}{n}}\left(\sum_{i=1}^{n} \tau_i\right) > \frac{1}{e}$

is a sufficient condition for the Oscillation of all solutions of the delay differential equation (10.0). Hence the above example, that is, example 3.01, is Oscillatory by inequality (36.0) and also satisfies the sufficient condition for the Oscillation of the delay differential equation (14.0). Example 3.02 is non-oscillatory by our condition, that is, equation (36.0) but is oscillatory by the condition of Gÿori and Ladas (1990). While example 3.03 is non-Oscillatory by the condition of Gÿori and Ladas (1991) and also by our condition equation (36.0)

Conclusion

It is clear from the above examples that both our condition, that is, inequality (36.0) and that of Gÿori and Ladas (1991) coincide for oscillatory equation for example 3.01, and non-oscillatory for example 3.02 by our condition (36.0). finally example 3.03 is non-oscillatory by our condition ant that of Gÿori and Ladas (1991).

References

Arino, O; Jawhari, A & Gÿori, I (1984). Oscillation Criteria in Delay Equations. *Journal of Differential Equations, 53, 115-123.*

Arino, O; Ladas, G & Sficas Y. G. (1987). On Oscillation of Some Retarded Differential Equations. *SIAM Journal of Mathematical Analysis, 18(1), 64-73*

Bamaina, M. M. (2004). Conditions for Oscillation and Non-oscillation of First Order Linear Delay Differential Equations, *Unpublished M.Sc. Project Abubakar Tafawa Balewa University.*

Sesay, M. S, Haruna, Y. & Bamaina, M. M(2005). Oscillatory and Non-oscillatory Conditions of First Order Linear Delay Differential Equations. *Science Forum; Journal of Pure and Applied Sciences, 8(1), 41-51.*

Grove, E. A; Ladas, G. & Schinas, J. (1988). Sufficient Conditions for the Oscillation of Delay and Neutral Equations. *Canad. Math. Bull, 31(4), 459-466.*

Gÿori, I. (1989). Oscillations of Retarded Differential Equations of the Neutral and the Mixed Types, *Journal of Mathematical Analysis and Applications, 141(1), 1-20.*

Gÿori, I; Ladas, G. & Pakula, L. (1989). On Oscillation of Unbounded Solutions. *Proceedings of the American Mathematical Society, 106(3), 785-792.*

Gÿori I; Ladas, G & Pakula, L (1990). Oscillation Theorems for Delay Differential Equations Via Transforms. *Canad. Math. Bull, 33(3), 323-326.*

Gÿori I. & Ladas G. (1991). Oscillation Theory of Delay Differential Equations with Applications. Clarendon Press Oxford.

Hunt, B. R. & Yorke, J. A. (1984). When all solution of $x' = -\sum_{i=1}^{\infty} q_i(t)x\big(t-\tau_i(t)\big)$ Oscillates. *Journal if Differential Equations, 53,139-145.*

Kusano, T. (1982). On Even Order Functional Differential Equations with Advanced and Retarded Arguments. *Journal Differential Equations, 45, 75-84.*

Ladas, G. (1979). Sharp Conditions for Oscillation Caused by Delays. *Applicable Anal., 9, 93-98.*

Ladas, G. & Stavroulakis, P. I. (1982a). On Delay Differential Inequalities of First Order. *Funcciaj Ekvacioj 25, 105-113.*

Ladas,G. & Stavroulakis, P. I. (1982b). Oscillations Caused by Retarded and Advanced Arguments. *Journal of Differential Equations, 44, 134-152.*

Ladas,G; Sficas, Y. G & Stavroulakis, P. I. (1983). Necessary and Sufficient Conditions for Oscillations, *American Mathematical Monthly, 90, 637-640.*

Ladas,G. & Sficas, Y. G. (1986). Oscillations of Higher Order Neutral Equations. *Journal Austral. Math. Soc. Ser. B, 27, 502-511.*

Ladde, G. S (1977). Oscillations Caused by Retarded Perturbations of First Order Linear Differential Equations. *Atti. Accad. Naz. Lincei Rend. Cl.Sci.Fis. Mat. Natur. 351-359.*

Öğûnmez H. & Öclan Ö. (2013). Oscillation of Higher Order Systems of Differential Equations. *International Journal of Mathematical Analysis, 7(15), 735-740.*

Philos, C. G. (1990). On the Oscillation of Differential Equations with Periodic Coefficients. *American Mathematical Society, 111(2), 433-441.*

Tramov, M. I. (1975). Conditions for Oscillatory Solutions of First Order Differential Equations with a Delayed Argument, *Isv.Vysš. Učebn. Zaved. Mat 19(3) 92-96.*

Wider, D. V. (1971). An Introduction to Transform Theory, Academic Press.